Love in the Time of Tindera

Gabriel García Marquez
We have a group of suitors and reviewers
Each has preferences over the other group.
We seek a stable matching between the two.
Unstable Matching
Unstable Matching

G1 prefers 2 to 3

B2 prefers 1 to 2
G1 prefers 2 to 3

B2 prefers 1 to 2

Unstable Matching
Stable Matching
Stable matching has many practical applications.
The National Resident Matching Program (NRMP) is a private, not-for-profit corporation established in 1952 to optimize the rank-ordered choices of applicants and program directors. The NRMP is not an application processing service; rather, it provides an impartial venue for matching applicants' and programs' preferences for each other consistently.

The first Main Residency Match® ("the Match") was conducted in 1952 when 10,400 internship positions were available for 6,000 U.S. graduating seniors. By 1973, there were 19,000 positions for just over 10,000 U.S. graduating seniors. Following the demise of internships in 1975, the number of first-year post-graduate (PGY-1) positions dropped to 15,700. The number of PGY-1 positions offered gradually increased through 1994 and then began to decline slowly until 1998. This year saw a record-high 26,678 PGY-1 positions offered (Figure 1), marking the twelfth consecutive annual increase in such positions.

The trend in the total number of applicants since 1952 is more dramatic, starting with 6,000 in 1952 and rising to a high of 36,056 in 1999. After a decline of 5,052 applicants from 1999 to 2003, the number of applicants has increased each year since the 2004 Match. Applicants registered for the 2014 Match reached an all-time high of 40,394, an increase of 59 applicants over 2013.

For more information about the NRMP, please visit: www.nrmp.org. Additional data and reports for the Main Residency Match and the Specialties Matching Service® (SMS®) can be found at: www.nrmp.org/match-data. Instructions on how to request NRMP data also are provided.

Figure 1: Applicants and 1st Year Positions in The Match, 1952 - 2014
## Table 1: Summary of Match Results

<table>
<thead>
<tr>
<th>Applicant Type</th>
<th>2013 Graduates</th>
<th>Prior Year Graduates(^1)</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>CMG</td>
<td>2571</td>
<td>74</td>
<td>2645</td>
</tr>
<tr>
<td>IMG</td>
<td>146</td>
<td>353</td>
<td>499</td>
</tr>
<tr>
<td>USMG</td>
<td>23</td>
<td>2</td>
<td>25</td>
</tr>
<tr>
<td><strong>TOTAL</strong></td>
<td><strong>2740</strong></td>
<td><strong>429</strong></td>
<td><strong>3169</strong></td>
</tr>
</tbody>
</table>

**Keywords:** Summary, match results, 1st and 2nd iteration combined, prior year, current year, unmatched, final participation

**Note 1:** Includes graduates from all years prior to 2013.
Definition: matchings

\[ M = \{ m_1, m_2, \ldots, m_n \} \]

\[ W = \{ s_1, \ldots, s_n \} \]

\[ S = \{ (m_i, s_i) \} \text{ such that each } m \text{ and each } w \text{ appear in exactly one pair in } S. \]
Definition: matchings

\[ M = \{m_1, \ldots, m_n\} \]
\[ W = \{w_1, \ldots, w_n\} \]
\[ S = \{(m_{i_1}, w_{j_1}), \ldots, (m_{i_k}, w_{i_k})\} \]

Each \( m_i \) (\( w_i \)) appears only one in a pairing. A matching is perfect if every \( m_i \) appears.
preferences
Definition: preferences

\[ M = \{m_1, \ldots, m_n\} \]

\[ w_i \preceq_{m_i} w_j \quad \text{"} m_i \text{ prefers } w_j \text{ to } w_i \text{"} \]
Example: preferences

\[ M = \{ m_1, \ldots, m_n \} \]

\( m_i \) has a preference relation \(<_m_i\) on the set \( W \)

\[ w_1 <_{m_i} w_4 <_{m_i} w_2 <_{m_i} w_8 \cdots w_n \]
\[ S = \left\{ \left( \begin{array} \text{dog} \\
\end{array} \right), \left( \begin{array} \text{pig} \\
\end{array} \right) \right\} \]

\((\text{pig}, \text{pig})\) is an instability
Def: instability

\[ S = \left\{ \left( \begin{array}{c} m \\ w \end{array} \right), \left( \begin{array}{c} w' \\ n' \end{array} \right) \right\} \]

is an unmatched pair \((m, w)\) such that

- \(m\) prefers \(w\) to its current match \(w'\)
- \(w'\) prefers \(m\) to its current match \(m'\)
Def: instability

\[ S = \left\{ \begin{array}{c}
\left( \begin{array}{c}
\text{bear}
\end{array} \right)
\left( \begin{array}{c}
\text{Harvard}
\end{array} \right)
\left( \begin{array}{c}
\text{NYU}
\end{array} \right)
\left( \begin{array}{c}
\text{SWU}
\end{array} \right)
\end{array} \right\}
\]

\[(m^*, w^*) \notin S\]

\[ w' <_{m^*} w^* \]

\[ m' <_{w^*} m^* \]
\[ M = \{ (s_1, r_1), (s_2, r_2), \ldots, (s_n, r_n) \} \]

is a stable matching if

No unmatched pair \((s^*, r^*)\) prefer each other to their partners in \(M\).
Example 2
Prove: for every input there exists a stable matching.
proposal algorithm

- Start with everyone unmatched.

While there is an unmatched suitor $S$

Let $r$ be highest ranked reviewer that $S$ hasn’t proposed to.

$S$ proposes a match with $r$.

If $r$ is unmatched or is matched to $(s', r)$ and $s' < r$, $S$ breaks the match $(s', r)$ & creates the match $(s, r)$. 

**StableMatch**\((M, W, \prec_m, \prec_w)\)

1. Initialize all \(m, w\) to be **free**
2. **while** \(\exists \text{FREE}(m)\) and hasn’t proposed to all \(W\)  
   3. **do** Pick such an \(m\)  
      4. Let \(w \in W\) be highest-ranked to whom \(m\) has not yet proposed  
      5. **if** \(\text{FREE}(w)\)  
         6. **then** Make a new pair \((m, w)\)  
      7. **elseif** \((m', w)\) is paired and \(m' \prec_w m\)  
         8. **do** Break pair \((m', w)\) and make \(m'\) free  
      9. **end if**  
   10. **end do**  
11. **return** Set of pairs
Proposal algorithm ends
Proposal algorithm ends

$O(n^2)$ steps

each $m$ proposes at most once to each $w$.

each $m$ proposes at most $n$ times.

size of $M$ is at most $n$. 
output is a matching

Each $m$ only appears at most once in the output. By lines 6 and 9, when a match is added to potential output, both parties are unmatched at the time of match by lines 2, 5 and/or 8.
**StableMatch**($M, W, \prec_m, \prec_w$)

1. Initialize all $m, w$ to be FREE
2. while $\exists \text{FREE}(m)$ and hasn’t proposed to all $W$
   
   do Pick such an $m$

   Let $w \in W$ be highest-ranked to whom $m$ has not yet proposed

   if FREE($w$)

   then Make a new pair $(m, w)$

   elseif $(m', w)$ is paired and $m' \prec_w m$

   do Break pair $(m', w)$ and make $m'$ free

   Make pair $(m, w)$

3. return Set of pairs
\textbf{StableMatch}(M, W, \prec_M, \prec_W)

1. Initialize all $m, w$ to be \text{FREE}

2. \textbf{while} $\exists \text{FREE}(m)$ and hasn’t proposed to all $W$

   \hspace{1em} \textbf{do} Pick such an $m$

   \hspace{2em} Let $w \in W$ be highest-ranked to whom $m$ has not yet proposed

   \hspace{3em} if $\text{FREE}(w)$

   \hspace{4em} then Make a new pair $(m, w)$

   \hspace{3em} elseif $(m', w)$ is paired and $m' \prec_W m$

   \hspace{4em} do Break pair $(m', w)$ and make $m'$ free

   \hspace{5em} Make pair $(m, w)$

3. \textbf{return} Set of pairs
\[ |M| = n. \quad \text{Because} \]

\[ \Rightarrow \text{if there is an unmatched writer} \]

\[ \Rightarrow \exists \text{an unmatched reviewer.} \]

(\text{so alg has not terminated yet})
output is perfect

if $\exists m$ who is free, then

$\exists w$ who has not been asked
output is stable

Proof: by contradiction. Since output is not stable, there exists an unmatched pair \((m^*, w^*)\) such that \(w \leq m^* w^*\) and \(m \leq w^* m^*\), and \((m^*, w) \not\in M\) and \((m, w^*) \not\in M\).
output is stable

spse not. \( \exists (m^*, w), (m, w^*) \in S \) \( w <_{m^*} w^* \) \( m <_{w^*} m^* \)

Consider the moment when \( w^* \) is matched with \( m^* \) and the moment when \( m^* \) is matched with \( w \).

1. \( m^* \) must have proposed to \( w \) last. But we know that \( m^* \) preferred \( w^* \) to \( w \). And by the algorithm, this means that \( m^* \) proposal to \( w^* \) before proposing to \( w \).

2. What happened when \( m^* \) proposed to \( w^* \)? \( @ (m^*, w^*) \) was made but then at some point \( (m, w^*) \) was made or \( w^* \) was already matched to \( m' \) and \( m^* <_{w^*} m' \).

In both cases, this suggests \( m^* <_{w^*} m \) which contradicts above.
output is stable

spse not.  \( \exists (m^*, w), (m, w^*) \in S \quad w \prec_{m^*} w^* \quad m \prec_{w^*} m^* \)

\( m^* \) last proposal was to \( w \)
but \( w \prec_{m^*} w^* \) and so \( m^* \) must have already asked \( w^* \)
and must have been rejected by \( m^* \prec_{w^*} m' \)
then either \( m' \prec_{w^*} m \) or \( m' = m \)
which contradicts assumption \( m \prec_{w^*} m^* \)
Proposer wins
Proposer wins
Remarkable theorem

w is valid for m: if \exists a stable matching S such that \((m,w) \in S\).

best(m): best(m) is valid for m and there is no valid \(w^*\) such that \(\text{best}(m) \leq_m w^*\)

Thm: G-S returns the match \(\exists (m, \text{best}(m))\) (propose optimal match).
**GS is Suitor-optimal.**

**Proof:** Suppose that GS did not return the $S^* = \{ (m, \text{best}(m))^\circ \}$, it returned $S \neq S^*$, i.e., there is some $m_j$ such that $w_j = \text{best}(m)$. Since $(m, w)$ was a valid match, $w$ must prefer $m$.

**Conclusion:** $S$ was not stable b/c of $(m, w)$.

⇒ contradiction to the underlined sentence.
GS matching vs R-opt
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Not honest
Not honest
Not honest
Guns and butter

\[
\begin{align*}
\max x + y \\
4x - y & \leq 8 \\
2x + y & \leq 10 \\
5x - 2y & \geq -2 \\
x, y & \geq 0
\end{align*}
\]
\begin{align*}
4x - y & \leq 8 \\
2x + y & \leq 10 \\
5x - 2y & \geq -2 \\
x, y & \geq 0
\end{align*}
$4x - y \leq 8$
$2x + y \leq 10$
$5x - 2y \geq -2$
$x, y \geq 0$
\[ \begin{align*}
4x - y & \leq 8 \\
2x + y & \leq 10 \\
5x - 2y & \geq -2 \\
x, y & \geq 0
\end{align*} \]
\[4x - y \leq 8\]
\[2x + y \leq 10\]
\[5x - 2y \geq -2\]
\[x, y \geq 0\]
Certificate of optimality

$$\max x + y$$

$$4x - y \leq 8$$
$$2x + y \leq 10$$
$$5x - 2y \geq -2$$
$$x, y \geq 0$$